Mathematics-II – X20001

(2nd Sem Civil PDDC 2013)

Assignment-1, Fourier Integral and Fourier Transform

(Based on Fourier Integral)

1. Find the (a) Fourier Cosine integral and (b) Fourier sine integral representation of $f(x) = \begin{cases} \sin x, & 0 \le x, \le \pi \\ 0 & x > \pi \end{cases}$.

2. Using Fourier integral representation, show that $\int_0^\infty \frac{\cos x\lambda + \lambda \sin x\lambda}{1+\lambda^2} d\lambda = \begin{cases} 0, & x < 0 \\ \frac{\pi}{2}, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$.

3. Using Fourier sine integral show that $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda d\lambda = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > 0 \end{cases}$.

4. Find the Fourier integral representation of the function $f(x) = \begin{cases} 2, & |x| < 2 \\ 0, & |x| > 2 \end{cases}$.

5. Prove that $\int_0^\infty \frac{\cos x\lambda}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}; \quad x \ge 0.$

6. Find the (a) Fourier Cosine integral and (b) Fourier sine integral representation of $f(x) = \begin{cases} e^x; & 0 < x < 1 \\ 0, & x > 1 \end{cases}$.

7. Find the Fourier integral representation of $f(x) = \begin{cases} \cos x; & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$.

8. Find the Fourier cosine integral of $f(x) = \begin{cases} x; & 0 < x < a \\ 0, & |x| > a \end{cases}$.

(Based on Fourier Transform)

1. Find the Fourier Transform of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0 & |x| > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

2. Find the Fourier Transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos \frac{x}{2} dx$.

3. Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$. deduce that $e^{-x^2/2}$ self reciprocal in respect of Fourier transform.

4. Find the Fourier cosine transform of e^{-x^2}.

5. Find the Fourier sine transform of $e^{|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m > 0$.

6. Find the Fourier cosine transform of $f(x) = \begin{cases} x; & 0 < x < 1 \\ 2-x; & 1 < x < 2 \\ 0; & x > 2 \end{cases}$.

7. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

8. Find the Fourier sine and cosine transform of $f(x) = \begin{cases} 1; & 0 < x < a \\ 0, & x > a \end{cases}$.

9. Find the Fourier transform of $f(x) = \begin{cases} x^2; & |x| < a \\ 0, & |x| > a \end{cases}$.

ASSIGNMENT : 1.

$\Longrightarrow$ (Based on fourier Transform)

Ex : 1 Find the fourier tranform of $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Hence evaluat $\int_0^\infty \frac{\sin x}{x}\, dx$.

soln : The fourier transform is

$$f(s) = \int_{-\infty}^{\infty} f(t)\, e^{2st}\, dt.$$

$$= \int_{-\infty}^{-1} 0\, e^{ist}\, dt \int_{-1}^{1} 1.\, e^{2st}\, dt + \int_{1}^{\infty} 0\, e^{2st}\, dt$$

$$= \int_{-1}^{1} e^{2st}\, dt$$

$$= \left[\frac{e^{2st}}{2s} \right]_{-1}^{1} = \left[\frac{e^{2s}}{2s} - \frac{e^{-2s}}{2s} \right]$$

$$= \frac{e^{2s} - e^{-2s}}{2s} = \frac{e^{2s} - e^{-2s}}{2s} \times \frac{2}{2} = \frac{2}{s} \sin s \qquad \text{—— (1)}$$

Now from inverse F.T of $f(s)$ is

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{s} \sin s\, e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\, e^{-isx}\, dx$$

Putting $x = 0$ then

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds \quad \therefore \quad f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin x}{x}\, dx \qquad \left(\because \frac{\sin x}{x} \text{ is an even fn}\right)$$

$$\therefore \int_0^\infty \frac{\sin x}{x}\, dx = \pi/2\, f(x)$$

$$= \begin{cases} \pi/2 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$$

Ex: 2 find the fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{\pi}{2}\, dx$

Soln :- $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$

$$= \int_{-\infty}^{-1} 0\, dt \int_{-1}^{1} (1-t^2)\, e^{ist}\, dt + \int_{1}^{\infty} 0\, dt$$

$$= \int_{-1}^{1} (1-t^2)\, e^{ist}\, dt$$

$$= \left[(1-t^2)\frac{e^{ist}}{is} - (-2t)\left(\frac{e^{ist}}{(is)^2}\right) + (-2)\left(\frac{e^{ist}}{(is)^3}\right) \right]_{-1}^{1}$$

$$= \left[\left\{ 0 - (-2\cdot 1)\frac{e^{is}}{i^2 s^2} + (-2)\frac{e^{is}}{i^3 s^3} \right\} - \left\{ 0 - (-2)(-1)\frac{e^{-is}}{i^2 s^2} + (-2)\frac{e^{-is}}{i^3 s^3} \right\} \right]$$

$$= \left[-2\frac{e^{is}}{s^2} + \frac{2e^{is}}{is^3} - 2\frac{e^{-is}}{s^2} - \frac{2e^{-is}}{is^3} \right]$$

$$= \left[-\frac{2}{s^2}\left(e^{is} + e^{-is}\right) + \frac{2}{is^3}\left(e^{is} - e^{-is}\right) \right]$$

$$= \frac{-4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} + e^{-is}}{2}\right)$$

$$= \frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s$$

$$= \frac{-4}{s^3}\left(s\cos s - \sin s\right) \qquad (\because \text{ ল.সা.অ})$$

Now by invers F.T $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}\left(s\cos s - \sin s\right) e^{-isx}\, dx = f(x)$$

$$= \begin{cases} 1-x^2 ; & |x| \le 1 \\ 0 ; & |x| > 1 \end{cases}$$

Putting $x = 1/2$; then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}(s\cos s - \sin s)\, e^{-\frac{is}{2}}\, ds = 1 - 1/4 = 3/4$$

$$-\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{s\cos s - \sin s}{s^3}\left(\cos s/2 - \sin s/2\right) ds = 3/4$$

$$(\because e^{i\theta} = \cos\theta - i\sin\theta)$$

$$\therefore \int_{-\infty}^{\infty}\left[\frac{s\cos s - \sin s}{s^3}\cdot\cos s/2 \cdot i\,\frac{s\cos - \sin s}{s^3}\sin\frac{s}{2}\right] ds = -\frac{3\pi}{8}$$

$$(\because (x,y) = x + iy)$$

$\therefore$ Equating real & imaginary part both side we get

$$\int_{-\infty}^{\infty} \frac{s\cos - \sin s}{s^3}\cdot\cos s/2\, ds = -\frac{3\pi}{8}$$

$$\therefore 2\int_{0}^{\infty} \frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\, ds = -\frac{3\pi}{8}$$

$$\therefore \int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\, dx = -\frac{3\pi}{16}$$

$$\left(\because \frac{s\cos - \sin s}{s^3}\cos s/2 \text{ is even fn.}\right)$$

Q:3 Find the fourier transform of $e^{-a^2x^2}$, $a > 0$. deduce that $e^{-x^2/2}$ self reciprocal in respect of fourier transform.

Soln:- $$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2v^2}\, e^{-i\lambda v}\, dv$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(av - \frac{i\lambda}{2a}\right)^2}\, e^{-\frac{\lambda^2}{4a^2}}\, dv$$

$$= \frac{e^{-\frac{\lambda^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(av - \frac{i\lambda}{2a}\right)^2} dv \qquad \text{Putting } av - \frac{i\lambda}{2a} = t$$

$$= \frac{e^{-\frac{\lambda^2}{4a^2}}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-\frac{\lambda^2}{4a^2}}}{a\sqrt{2\pi}} \cdot \sqrt{\pi}, \qquad \text{since } \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$\therefore F(\lambda) = \frac{e^{-\frac{\lambda^2}{4a^2}}}{a\sqrt{2}}$$

(2) Replace a^2 by $\frac{1}{2}$ in (1).

(3) If $a = \frac{1}{\sqrt{2}}$ in (1) we get $f\left(e^{-\frac{\lambda^2}{2}}\right) = e^{-\frac{\lambda^2}{2}} = e^{-\frac{x^2}{2}}$

($\because$ Replacing $\lambda = x$)

$\therefore$ We have shown above that $e^{-\frac{x^2}{2}}$ is self-reciprocal under fourier transform.

$\therefore$ Defn of self-reciprocal :- If the transform of $f(x)$ is $f(\lambda)$, the function $f(x)$ is called self-reciprocal.

Q24: find the fourier cosine transform of e^{-x^2}.

Soln :- $$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2} \cos\lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\lambda^2} (-\cos\lambda x + \lambda \sin\lambda x) \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{e^{-\infty}}{1+\lambda^2} (-\cos\lambda\infty + \lambda \sin\lambda\infty) \right\} - \left\{ \frac{e^{0}}{1+\lambda^2} (-\cos\lambda 0 + \lambda \sin\lambda \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} + \left\{ \frac{1}{1+\lambda^2} + 0 \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\lambda^2} \right)$$

Ex. 5 find the fourier sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi e^{-m}}{2}$, $m>0$

Soln :- The fourier sine transform is

$$f_s(s) = \int_0^\infty f(x) \sin sx \, dx \quad \text{—— (1)}$$

x beeing +ve in the integral $(0, \infty)$

So, $e^{-|x|} = e^{-x}$ $\therefore$ from (1)

$$S_s(s) = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2+s^2} (-\sin sx - s\cos sx) \right]_0^\infty$$

$$= \left[\{0\} - \left\{ \frac{1}{1+s^2} \right\} \{(0-s)\} \right. = \frac{s}{1+s^2} \quad \text{—— (2)}$$

Now $f(x) = \frac{2}{\pi} \int_0^\infty f_s(s) \sin sx \, ds$.

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

$$\therefore = \int_0^\infty \frac{s \sin sx}{1+s^2} ds = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-x}$$

changing from x to m

$$\therefore \int_0^\infty \frac{s \sin sm}{1+s^2} ds = \frac{\pi}{2} e^{-m}$$ and changing s to x

$$\therefore \int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

Ex: 6 Find the fourier cosine transform of

$$f(x) = \begin{cases} x ; & 0<x<1 \\ 2-x ; & 1<x<2 \\ 0 ; & x>0 \end{cases}$$

$$f_c(s) = \int_0^\infty f(t) \cos sx \, dx$$

$$= \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^\infty 0 \cos sx \, dx$$

$$= \left[x\left(\frac{\sin sx}{s}\right) - (1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^1 + \left[(2-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_1^2 + 0$$

$$= \left[\left\{ \frac{\sin s}{s} + \frac{\cos s}{s^2} \right\} - \left\{ 0 + \frac{1}{s^2} \right\} \right] + \left[\left\{ 0 - \frac{\cos 2s}{s^2} \right\} - \left\{ \left(\frac{\sin s}{s}\right) - \frac{\cos s}{s^2} \right\} \right]$$

$$= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}$$

Ex: 7 Find the fourier sine transform of $\frac{e^{-ax}}{x}$.

Soln: $f_s(s) = \int_0^\infty f(x) \sin sx \, dx$

$$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = f(s) \text{ ; say} \quad \text{---(1)}$$

Diff. eqn (1) with r.t. s both side.

$$\frac{dF}{ds} = \int_0^\infty \frac{d}{ds}\left(\frac{e^{-ax}}{x} \sin sx\right) dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} (x \cos sx) \, dx$$

$$= \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2+s^2} \left(-a\cos sx + s\sin sx \right) \right]_0^\infty$$

$$= \left[0 - \left\{ \frac{1}{a^2+s^2} \left(-a(1) + s(0) \right) \right\} \right]$$

$$\frac{df}{ds} = \frac{a}{s^2+a^2}$$ Int. both side w.r.t. s, we get

$$\therefore f = \int \frac{a}{s^2+a^2}\, ds + c$$

$$= a \int \frac{1}{s^2+a^2}\, ds + c$$

$$= a \frac{1}{a} \tan^{-1} s/a + c$$

$$f(s) = \tan^{-1} s/a + c \quad \text{—— (2)}$$

Taking $s=0$ in —— (1)

$\therefore f(s) = 0$, Also $f(s) = 0$ when $s=0$

$\therefore c = 0$

$\therefore$ from (2) $f(s) = \tan^{-1} s/a$

Q:8 find the fourier sine and cosine transform of

$$f(x) = \begin{cases} 1 : & 0 < x < a \\ 0, & x > a \end{cases}$$

Soln:- from the definations of cosine and sine transform of $f(x)$ we have

$$f_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\lambda x \, dx \qquad (\because \text{cosine formula})$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a 1 \cdot \cos\lambda x \, dx + \int_a^\infty 0 \cdot \cos\lambda x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos\lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\lambda x}{\lambda} \right]_0^a = \sqrt{\frac{2}{\pi}} = \left[\frac{\sin a\lambda}{\lambda} \right]$$

$$f_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\lambda x \, dx. \qquad (\because \text{from formula})$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 1 \sin\lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos\lambda x}{\lambda} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\left\{ \frac{\cos\lambda a}{\lambda} \right\} - \left\{ \frac{\cos\lambda 0}{\lambda} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos\lambda a}{\lambda} + 1 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos a\lambda}{\lambda} \right]$$